

CURVES IN THE DOUBLE PLANE

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ABSTRACT. We study in detail locally Cohen-Macaulay curves in \mathbb{P}^3 which are contained in a double plane $2H$, thus completing the classification of curves lying on surfaces of degree two. We describe the irreducible components of the Hilbert schemes $H_{d,g}(2H)$ of locally Cohen-Macaulay curves in $2H$ of degree d and arithmetic genus g , and we show that $H_{d,g}(2H)$ is connected. We also discuss the Rao module of these curves and liaison and biliaison equivalence classes.

1. INTRODUCTION

Much attention has been given in recent years to the classification of curves in projective space. Here we define a curve to be a purely one-dimensional locally Cohen-Macaulay (i.e. without embedded points) closed subscheme of \mathbb{P}_k^3 , the projective three-space over an algebraically closed field k . Our goal in this paper is to answer all the interesting questions about curves contained in surfaces of degree two. We succeed, with one notable exception.

Curves in the nonsingular quadric surface Q and in the quadric cone Q_0 are well-known [6] (Exercises III 5.6 and V 2.9). Curves in the union of two planes $H_1 \cup H_2$ were studied in [7], section 5. The main contribution of this paper is the systematic study of curves contained in a double plane $2H$.

We are also interested in flat families of curves, so that we can identify the irreducible components of the Hilbert scheme. Families of curves whose general member lies on a nonsingular quadric Q and whose special member lies on a cone Q_0 or the union of two planes $H_1 \cup H_2$ were studied in [8]. In this paper we will identify the irreducible components of the scheme $H_{d,g}(2H)$ of curves of degree d and genus g lying in a double plane, and specializations between them. The question we have not been able to answer is what families are there whose general member lies on a nonsingular quadric and whose special

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member lies in $2H$? For example, we do not know if there is a family specializing from four skew lines on a quadric to a curve contained in the double plane.

Now let us describe the contents of the current paper. To each curve C contained in the double plane $2H$, we assign a triple $\{Z, Y, P\}$ consisting of a zero-dimensional closed subscheme Z in the (reduced) plane H , and two curves Y, P in H . Roughly speaking, P with embedded points Z is the intersection of C with H , while Y is the residual intersection (Proposition 2.1). Conversely to each such triple, satisfying certain conditions, we can associate curves C in $2H$ (Proposition 3.1). The numerical invariants of C can be computed in terms of Z, Y, P (section 6). Thus the triple $\{Z, Y, P\}$ of data in the plane H is an effective tool for studying curves C in the double plane.

We can translate information about a flat family of curves in $2H$ to families of triples in H (section 4). This allows us to identify the irreducible components of the Hilbert scheme $H_{d,g}(2H)$. They are given by triples of integers (z, y, p) representing the length of Z and the degrees of Y and P (Theorem 5.1).

We also discuss the Rao module of these curves (section 6) and liaison and biliaison equivalence classes (section 7). Finally, generalizing a technique of Nollet [16], we show the existence of flat families joining the various irreducible components of $H_{d,g}(2H)$, and conclude that $H_{d,g}(2H)$ is connected (Theorem 8.1).

Our purpose in studying curves in the double plane was to treat one very special case in view of the more general problem whether the Hilbert scheme $H_{d,g}$ of all locally Cohen-Macaulay curves in \mathbb{P}^3 is connected for all d, g . Because of our connectedness result (8.1), it would be sufficient for that more general problem to show that any curve in \mathbb{P}^3 can be connected by a sequence of flat families to a curve in $2H$.

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2. THE TRIPLE ASSOCIATED TO A CURVE IN $2H$

We first recall the notion of “residual scheme” ([5], 9.2.8, and [18]). Suppose $T \subseteq W$ are closed subschemes of an ambient scheme V . The residual scheme R of T in W is the closed subscheme of V with ideal sheaf

$$\mathcal{I}_R = (\mathcal{I}_W : \mathcal{I}_T).$$

Intuitively, R is equal to W minus T . It does not depend on the ambient scheme. We will need this notion in the following two cases.

If T is a Cartier divisor in V , then $\mathcal{I}_R \mathcal{I}_T = \mathcal{I}_W$. On the other hand, suppose that H is an effective Cartier divisor on V , and T is the scheme theoretic intersection of W and H . In this case, we say (cf. [9], p.176) that R is the residual scheme to the intersection of W with H . We have

$$\mathcal{I}_R \mathcal{I}_H = \mathcal{I}_W \cap \mathcal{I}_H$$

hence an exact sequence:

$$(1) \quad 0 \rightarrow \mathcal{I}_R(-H) \rightarrow \mathcal{I}_W \rightarrow \mathcal{I}_{W \cap H, H} \rightarrow 0.$$

Let H be a plane in \mathbb{P}_k^3 , defined by $h = 0$. Let C be a curve contained in the scheme $2H$ (“the double plane”) defined by the equation $h^2 = 0$. To the curve C we will associate a triple $T(C) = \{Z, Y, P\}$ where Z is a zero-dimensional subscheme of H , and Y, P are curves in H , with $Z \subseteq Y \subseteq P$.

First consider the scheme-theoretic intersection $C \cap H$. This will be a one dimensional subscheme of H , possibly with embedded points. So we can write

$$\mathcal{I}_{C \cap H, H} = \mathcal{I}_{Z, H}(-P)$$

where Z is zero-dimensional and P is a curve. In fact, the inclusion $\mathcal{I}_{C \cap H, H} \hookrightarrow \mathcal{O}_H$ defines a global section of the invertible sheaf $\mathcal{H}om(\mathcal{I}_{C \cap H, H}, \mathcal{O}_H)$ whose scheme of zeros is the effective Cartier divisor $P \subset H$, and Z is the residual scheme of P in $C \cap H$.

Next we let Y be the residual scheme to the intersection of C with H : it is a curve in \mathbb{P}^3 . By the discussion above we have an exact sequence

$$(2) \quad 0 \rightarrow \mathcal{I}_{Y, \mathbb{P}^3}(-1) \xrightarrow{h} \mathcal{I}_{C, \mathbb{P}^3} \rightarrow \mathcal{I}_{C \cap H, H} \rightarrow 0.$$

Since C is contained in $2H$, Y will be contained in H . Furthermore, since P is the largest curve in H which is contained in C , it is clear that $Y \subseteq P$. Note that C is contained in the reduced plane H if and only if Y is empty.

Now we use the inclusion $P \subseteq C \cap H$ to create a diagram as follows

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{I}_{Y, \mathbb{P}^3}(-1) & \xrightarrow{h} & \mathcal{I}_{C, \mathbb{P}^3} & \longrightarrow & \mathcal{I}_{C \cap H, H} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 (3) \quad 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{h} & \mathcal{I}_{P, \mathbb{P}^3} & \longrightarrow & \mathcal{I}_{P, H} \longrightarrow 0 \\
 & & \downarrow & & \downarrow u & & \downarrow \\
 0 & \longrightarrow & \mathcal{O}_Y(-1) & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{O}_Z(-p) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where p is the degree of P . Since C is locally Cohen-Macaulay, the ideal sheaf $\mathcal{I}_{C, \mathbb{P}^3}$ has depth ≥ 2 at every closed point. Therefore the sheaf \mathcal{L} , defined as the quotient $\mathcal{I}_P/\mathcal{I}_C$, will have depth ≥ 1 . Multiplying a section of \mathcal{L} by a local section of \mathcal{I}_Y will give something zero outside of Z . But \mathcal{L} can have no section supported at a point because of its depth. So we see that \mathcal{L} is an \mathcal{O}_Y -module. Because Y is a Gorenstein curve, \mathcal{L} is a reflexive sheaf of rank one [7], 1.6, and then by [7], 2.8 and 2.9, \mathcal{L} is of the form $\mathcal{L}(D-1)$ for some effective generalized divisor D on Y . Then by [7], 2.10, the quotient of \mathcal{L} by $\mathcal{O}_Y(-1)$ is

$$\mathcal{O}_Z(-p) \cong \omega_D \otimes \omega_Y^\vee(-1),$$

so

$$\mathcal{O}_Z(-p) \cong \omega_D(2-y)$$

where y is the degree of Y .

From this we conclude several things. First $Z = D$, so that $Z \subseteq Y$. Secondly, \mathcal{O}_Z is locally isomorphic to ω_Z , so Z is a Gorenstein scheme. Since Z is of codimension two in H , it follows from the theorem of Serre [4], Corollary 21.20, that Z is a locally complete intersection in H . Summing up,

Proposition 2.1. *To each curve C in $2H$, we can associate a triple $T(C) = \{Z, Y, P\}$ where $Z \subseteq Y \subseteq P \subset H$, and Z is a locally complete intersection zero-dimensional subscheme, and Y, P are curves. If we denote by g the arithmetic genus of C , by d, y, p the degrees of C, Y, P*

respectively, and by z the length of Z , then

$$\begin{aligned} d &= y + p \\ g &= \frac{1}{2}(y-1)(y-2) + \frac{1}{2}(p-1)(p-2) + y - z - 1. \end{aligned}$$

The computation of the degree and genus of C is straightforward from the exact sequence (2).

Remark 2.2. The curve C is arithmetically Cohen-Macaulay if and only if Z is empty. From the exact sequence (2) we obtain an exact sequence

$$0 \rightarrow H_*^1(\mathcal{I}_C) \rightarrow H_*^1(\mathcal{I}_{Z,H}(-p)) \rightarrow H_*^2(\mathcal{I}_Y(-1))$$

where, for a coherent sheaf \mathcal{F} on \mathbb{P}^3 , we let

$$H_*^i(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} H^i(\mathcal{F}(n)).$$

If Z is empty, the middle term is zero, and that forces the first term, which is the Rao module M_C of C to be zero also. Hence C is ACM.

Conversely, if $M_C = 0$, then we look at the particular twist

$$0 \rightarrow H^1(\mathcal{I}_{Z,H}(-1)) \rightarrow H^2(\mathcal{I}_Y(p-2)).$$

The last group is isomorphic to $H^1(\mathcal{O}_Y(p-2))$, which is zero because $p \geq y$. Hence $\text{length } Z = \dim H^1(\mathcal{I}_{Z,H}(-1)) = 0$, and Z is empty.

3. EXISTENCE OF CURVES WITH A GIVEN TRIPLE

Suppose given $\{Z, Y, P\}$ as in 2.1. Let \mathcal{L} be the sheaf $\mathcal{O}_Y(Z-1)$ associated to the generalized divisor Z on Y as in [7], 2.8. Then we will show how to construct a surjective map $u : \mathcal{I}_{P,\mathbb{P}^3} \rightarrow \mathcal{L}$, so as to define $\mathcal{I}_{C,\mathbb{P}^3} = \text{Ker } u$ as in the diagram (3). Note that C will be locally Cohen-Macaulay and pure dimensional because at closed points \mathcal{L} has depth one and $\mathcal{I}_{P,\mathbb{P}^3}$ has depth two.

The map u needs to be compatible with the existing inclusions $\mathcal{O}_{\mathbb{P}^3}(-1) \hookrightarrow \mathcal{I}_{P,\mathbb{P}^3}$ and $\mathcal{O}_Y(-1) \hookrightarrow \mathcal{L}$. Furthermore, \mathcal{L} is an \mathcal{O}_Y -module, so u must factor through $\mathcal{I}_{P,\mathbb{P}^3} \otimes \mathcal{O}_Y$. Tensor the middle line of the diagram (3) with \mathcal{O}_Y . Then it will be sufficient to define a surjective map \bar{u} to complete the diagram

$$(4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_Y(-1) & \longrightarrow & \mathcal{I}_P/\mathcal{I}_Y\mathcal{I}_P & \longrightarrow & \mathcal{O}_Y(-p) \longrightarrow 0 \\ & & \downarrow & & \downarrow \bar{u} & & \downarrow w \\ 0 & \longrightarrow & \mathcal{O}_Y(-1) & \longrightarrow & \mathcal{L} & \longrightarrow & \omega_Z(2-y) \longrightarrow 0 \end{array}$$

Since P is a complete intersection in \mathbb{P}^3 , the top row, which is the restriction to Y of the conormal sequence of P , splits. Thus to find \bar{u} ,

we need only find a map $v : \mathcal{O}_Y(-p) \rightarrow \mathcal{L}$ whose image in $\omega_Z(2-y)$ is surjective. Since Z is a locally complete intersection in H , the sheaf ω_Z is generated by a single element at every point, so any sufficiently general $w : \mathcal{O}_Y(-p) \rightarrow \omega_Z(2-y)$ will be surjective. To show that w lifts to v , think of $v \in H^0(Y, \mathcal{L}(p))$. Then we have an exact sequence

$$0 \rightarrow H^0(\mathcal{O}_Y(p-1)) \rightarrow H^0(\mathcal{L}(p)) \rightarrow H^0(\omega_Z(2+p-y)) \rightarrow H^1(\mathcal{O}_Y(p-1))$$

Now $p \geq y$, so $p-1 > y-3$ hence $H^1(\mathcal{O}_Y(p-1)) = 0$. Thus any surjective w lifts to v and gives the desired \bar{u} . We can now define C by setting $\mathcal{I}_{C, \mathbb{P}^3} = \text{Ker } u$. Since $\mathcal{L} \cong \mathcal{I}_P/\mathcal{I}_C$ has depth ≥ 1 at closed points, C is locally Cohen-Macaulay, and it is clear from our construction that $T(C) = \{Z, Y, P\}$.

Finally, we claim that the construction above gives a one to one correspondence between curves $C \subset 2H$ with $T(C) = \{Z, Y, P\}$ and global sections $v \in H^0(Y, \mathcal{L}(p))$ whose image in $H^0(\omega_Z(2+p-y))$ generates $\omega_Z(2+p-y)$ at every point. Note that it is enough to show that a surjective \bar{u} fitting in the diagram (4) is determined by its kernel. Now, if \bar{u}_1 and \bar{u}_2 have the same kernel and fit in the diagram (4), then their difference factors through a morphism $\omega_Z(2-y) \rightarrow \mathcal{L}$ which must be zero, thus $\bar{u}_1 = \bar{u}_2$. So we have:

Proposition 3.1. *For each triple $\{Z, Y, P\}$ as in 2.1 there exists a curve $C \subset 2H$ with $T(C) = \{Z, Y, P\}$. The set of such C is parametrized by an open subset of the vector space $H^0(Y, \mathcal{L}(p))$, which has dimension*

$$h^0(\mathcal{L}(p)) = z + (p-1)y + 1 - \frac{1}{2}(y-1)(y-2).$$

4. FAMILIES OF CURVES IN THE DOUBLE PLANE

In order to understand the Hilbert scheme of curves in the double plane, we must carry out the constructions of section 2 and 3 for flat families. Consider a base scheme S , and let $\mathcal{C} \subset 2H \times S$ be a family of curves of degree d and genus g , i.e. a closed subscheme, flat over S , whose fibre over each point $s \in S$ is a locally Cohen-Macaulay curve C_s in $2H$ of given degree and genus. The functor which to each S associates the set of such flat families is represented by a Hilbert scheme which we denote by $H_{d,g}(2H)$.

If $\mathcal{C} \subset 2H \times S$ is a flat family as above, the intersection $\mathcal{C} \cap (H \times S)$ need not be flat. For example, if S is integral, flatness of $\mathcal{C} \cap (H \times S)$ is equivalent to local constancy of the integers z, y, p associated to the fibres C_s as in section 2 above. Applying Mumford's flattening stratification to $\mathcal{C} \cap (H \times S)$, where \mathcal{C} is the universal family over $H_{d,g}(2H)$, we find that the scheme $H_{d,g}(2H)$ is stratified by locally

closed subschemes $H_{z,y,p}(2H)$ representing families for which $\mathcal{C} \cap (H \times S)$ is flat and the fibres correspond to z, y, p as above.

We claim that the procedures of section 2 and 3 above can be relativized over a base scheme S , to show that a flat family $\mathcal{C} \subset 2H \times S$ with the condition that $\mathcal{C} \cap (H \times S)$ is also flat gives rise to a triple $\{Z, Y, P\}$ where $Z \subseteq Y \subseteq P$ are closed subschemes of $H \times S$, flat over S , and where the fibres of Z are zero-dimension locally complete intersection subschemes of H , and the fibres of Y and P are curves in H . Conversely, given any such triple $\{Z, Y, P\}$, there are families of curves giving rise to this triple, and the resulting curves are parametrized by a global section of the sheaf $\mathcal{L}(P)$, where \mathcal{L} is the sheaf $\mathcal{H}om(\mathcal{I}_{Z,Y}, \mathcal{O}_Y(-1))$ and P is the divisor on $H \times S$ corresponding to P .

Thus there is a natural map from the scheme $H_{z,y,p}(2H)$ to the Hilbert flag scheme $D_{z,y,p}(H)$ which parametrizes triples of flat families $Z \subseteq Y \subseteq P$ in $H \times S$, flat over S , as above, whose fibres Z_s are locally complete intersection of length z , and Y_s and P_s are curves of degree y and p respectively. The fibres of $H_{z,y,p}(2H)$ over $D_{z,y,p}(H)$ are open subsets of the vector spaces $H^0(Y_s, \mathcal{L}_s(p))$, which have constant dimension, and thus $H_{z,y,p}(2H)$ appears as an open subscheme of a geometric vector bundle over $D_{z,y,p}(H)$.

The functorial details to justify all this are standard, if rather lengthy, so we leave them to the reader and content ourselves with summarizing the results in the following proposition.

Proposition 4.1. *The Hilbert scheme $H_{d,g}(2H)$ of curves in $2H$ of degree d and genus g is stratified by locally closed subschemes $H_{z,y,p}(2H)$ corresponding to families of curves whose integers (z, y, p) associated by 2.1 are constant.*

The scheme $H_{z,y,p}$ has a natural map to the Hilbert flag scheme $D_{z,y,p}(H)$ of triples $\{Z, Y, P\}$ as in 2.1, and this map makes $H_{z,y,p}(2H)$ into an open subset of a geometric vector bundle \mathcal{E} over $D_{z,y,p}(H)$, where \mathcal{E} is locally free of rank

$$z + (p - 1)y + 1 - \frac{1}{2}(y - 1)(y - 2).$$

Next, we study the Hilbert flag scheme $D_{z,y,p}(H)$.

Proposition 4.2. *Given integers z, y, p with $z \geq 0$, $p \geq y \geq 1$, the Hilbert flag scheme $\bar{D}_{z,y,p}(H)$ of closed subschemes $Z \subseteq Y \subseteq P$ in H , with Z of dimension zero and length z , and Y, P curves of degree y, p respectively, is irreducible and generically smooth of dimension*

$$z + \frac{1}{2}y(y + 3) + \frac{1}{2}(p - y)(p - y + 3).$$

Proof. Since $Y \subseteq P$, we can write $P = Y + W$, where W is an effective divisor, i.e. a curve, of degree $p - y$, which can be chosen independently of Z, Y . So

$$\bar{D}_{z,y,p}(H) = D_{z,y}(H) \times D_{p-y}(H),$$

where D_{p-y} is a projective space of dimension $\frac{1}{2}(p-y)(p-y+3)$ parametrizing curves of degree $p-y$. Thus we reduce to considering $D_{z,y}(H)$. This is the non-trivial part, which was proved by Brun and Hirschowitz [3], proposition 3.2. The main ideas of their proof are as follows.

We regard $D_{z,y}$ as a closed subscheme of $M \times K$, where $M = \text{Hilb}^z(H)$ is the scheme of zero dimensional closed subschemes of H of length z , and K is a projective space of dimension $\frac{1}{2}y(y+3)$ parametrizing curves of degree y in H . Let Z and Y denote the universal families over M and K respectively. It is a theorem of Fogarty [11], example 4.5.10, that M is smooth irreducible of dimension $2z$. Brun and Hirschowitz first show that $D_{z,y}$ is the zero scheme of a section of the rank z vector bundle $\mathcal{H}om(\mathcal{I}_Y(y), \mathcal{O}_Z(y))$ on $M \times K$, and so has codimension at most z at every point. Thus the dimension of $D_{z,y}$ is at least $z + \frac{1}{2}y(y+3)$ at every point. Next, observe that $D_{z,y}$ contains an open subset U corresponding to divisors of degree z on smooth curves, that is smooth irreducible of dimension $z + \frac{1}{2}y(y+3)$. To complete the proof, they use the theorem of Briançon [2] that the punctual Hilbert scheme of zero dimensional schemes of length z at a point has dimension $z - 1$. This shows that the fibre of $D_{z,y}$ over any curve Y has total dimension at most z . Thus the dimension of $D_{z,y} \setminus U$ must be strictly less than $z + \frac{1}{2}y(y+3)$, and hence it is contained in the closure of U . \square

Corollary 4.3. *Suppose $z \geq 0$ and $p \geq y \geq 1$, or $z = y = 0$ and $p \geq 1$. Then the scheme $H_{z,y,p}(2H)$ is irreducible and generically smooth of dimension*

$$2z + \frac{1}{2}y(y+1) + \frac{1}{2}p(p+3).$$

5. IRREDUCIBLE COMPONENTS OF $H_{d,g}(2H)$

We can now describe the irreducible components of $H_{d,g}(2H)$.

Theorem 5.1. *Let d and g be integers, with $d \geq 1$. $H_{d,g}(2H)$ is non-empty if and only if either $g = \frac{1}{2}(d-1)(d-2)$, or $d \geq 2$ and $g \leq \frac{1}{2}(d-2)(d-3)$.*

The irreducible components of $H_{d,g}$ are the closures $\bar{H}_{z,y,p}$ of the subschemes $H_{z,y,p}$ defined in section 4, where (z, y, p) varies in the set

of triples of nonnegative integers satisfying the following conditions:
 $p \geq 1$, $p \geq y$, $z = 0$ if $y = 0$, and

$$p = d - y$$

$$z = \frac{1}{2}(y-1)(y-2) + \frac{1}{2}(p-1)(p-2) + y - g - 1.$$

Proof. By corollary 4.3, $\bar{H}_{z,y,p}$ is irreducible of dimension $2z + \frac{1}{2}y(y+1) + \frac{1}{2}p(p+3)$. To see that $\bar{H}_{z,y,p}$ is an irreducible component, it is enough to observe that its generic point is not contained in a different $\bar{H}_{z',y',p'}$. Now this is clear, since the support of the generic point of $\bar{H}_{z,y,p}$ is the union of two smooth plane curves of degrees y and $p-y$ respectively.

By propositions 2.1 and 3.1, as a topological space $H_{d,g}$ is the disjoint union of its subschemes $H_{z,y,p}$, where z,y,p are nonnegative integers satisfying $p \geq 1$, $p \geq y$, $z = 0$ if $y = 0$, and

$$p = d - y$$

$$z = \frac{1}{2}(y-1)(y-2) + \frac{1}{2}(p-1)(p-2) + y - g - 1.$$

It remains to determine the triples (z, y, p) satisfying these conditions. We must have

$$z = \frac{1}{2}(d-2)(d-3) - g - (y-1)(d-y-2).$$

We now impose the conditions on z , y and p . The conditions $p \geq y$ and $p \geq 1$ translate into $y \leq d/2$ and $d-y-1 \geq 0$. As y and z must be nonnegative, we have

$$\frac{1}{2}(d-1)(d-2) - g = y(d-y-1) + z \geq 0$$

with equality if and only if $y = z = 0$ or $z = 0$, $y = 1$ and $d = 2$. In all other cases we have $y \geq 1$ and either $d-y > 1$, or $y = 1, d = 2$ and $z \geq 1$, hence

$$\frac{1}{2}(d-2)(d-3) - g = (y-1)(d-y-2) + z \geq 0.$$

□

Remark 5.2. How many irreducible components does $H_{d,g}$ have? If $d \neq 2$ and $g = 1/2(d-1)(d-2)$, $H_{d,g} = H_{0,0,d}$ is irreducible. $H_{2,0}$ has two irreducible components, namely $\bar{H}_{0,1,1}$ and $\bar{H}_{0,0,2}$.

Suppose $d \geq 2$ and $g \leq 1/2(d-2)(d-3)$ with $(d, g) \neq (2, 0)$. Let y_M be largest integer n in the closed interval $[1, d/2]$ such that $(n-1)(d-n-2) \leq 1/2(d-2)(d-3) - g$. The irreducible components

of $H_{d,g}$ are in one to one correspondence with the integers y in the closed interval $[1, y_M]$.

6. NUMERICAL INVARIANTS

Out of the exact sequence (2) one immediately computes the postulation of the curve C in terms of the two integers y and p and of the postulation of Z . However to compute the dimensions of the first and second cohomology modules of $\mathcal{I}_C(n)$ one needs some additional argument. For example, one could use the results on liaison of the next section. Here we take a different approach.

Choose a point R of $\mathbb{P}^3 \setminus H$, and let $\pi : 2H \rightarrow H$ be the morphism induced by the projection from R . π is a finite and flat morphism, and $\pi_*(\mathcal{O}_{2H}) = \mathcal{O}_H \oplus \mathcal{O}_H(-1)$. Now given any other point R' in $\mathbb{P}^3 \setminus H$, there is an automorphism of \mathbb{P}^3 which sends R to R' and fixes every point of H . It follows that for any coherent sheaf of \mathcal{O}_{2H} -modules \mathcal{F} , the isomorphism class of the sheaf $\pi_*\mathcal{F}$ is independent of the choice of the point R .

Proposition 6.1. *Let $\pi : 2H \rightarrow H$ be the morphism induced by the projection from R . Let C be a curve in $2H$. Then $\mathcal{E} = \pi_*(\mathcal{I}_{C,2H})$ is a locally free \mathcal{O}_H -module of rank two. Furthermore, the Rao module $H_*^1(\mathbb{P}^3, \mathcal{I}_C)$ is isomorphic to $H_*^1(H, \mathcal{E})$ as a module over the homogeneous coordinate ring of \mathbb{P}^3 .*

Proof. We use the fact proven in 2.1 that the sheaf of \mathcal{O}_Y -modules \mathcal{L} , defined as the quotient $\mathcal{I}_P/\mathcal{I}_C$ has depth ≥ 1 at closed points. It follows from this and the exact sequence

$$0 \rightarrow \pi_*(\mathcal{I}_{C,2H}) \rightarrow \pi_*(\mathcal{I}_{P,2H}) \rightarrow \mathcal{L} \rightarrow 0$$

that $\pi_*(\mathcal{I}_{C,2H})$ is locally free of rank two provided $\pi_*(\mathcal{I}_{P,2H})$ is locally free of rank two. But this is clear as we have an exact sequence

$$0 \rightarrow \mathcal{O}_H(-1) \rightarrow \pi_*(\mathcal{I}_{P,2H}) \rightarrow \mathcal{I}_{P,H} \cong \mathcal{O}_H(-p) \rightarrow 0.$$

Finally, from the exact sequence (2) we see that the Rao module of C is annihilated by the equation of H , hence it is isomorphic to $H_*^1(H, \pi_*(\mathcal{I}_{C,2H}))$ as a modules over $H_*^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3})$. \square

Corollary 6.2. *Let C be a curve of degree d in the double plane, with Rao module M_C . Let M_C^* denote the k -dual module to M_C (see [12], 0.1.7 p.20). Then $M_C^* \cong M_C(d-2)$, and in particular $h^1(\mathcal{I}_C(n)) = h^1(\mathcal{I}_C(d-2-n))$ for all integers n .*

Proof. Let $M = M_C \cong H_*^1(H, \mathcal{E})$. Since \mathcal{E} is a rank two vector bundle on H , we have $\mathcal{E}^\vee \cong \mathcal{E}(-c_1)$ where c_1 denotes the first Chern class of \mathcal{E} . By Serre duality we have $M^* \cong M(-c_1 - 3)$. Now from the exact sequence

$$(5) \quad 0 \rightarrow \mathcal{O}_H(-y-1) \rightarrow \mathcal{E} \rightarrow \mathcal{I}_{Z,H}(-p) \rightarrow 0.$$

we compute $c_1 = -p - y - 1 = -d - 1$, and we are done. \square

Remark 6.3. If we apply π_* to the exact sequence (2), we obtain an extension class in $\text{Ext}_H^1(\mathcal{I}_{Z,H}(-p), \mathcal{O}_H(-y-1))$, hence an element w in $H^0(\omega_Z(2+p-y))$. In the notation of section 3, one verifies that w is the image of the global section v of $\mathcal{L}(p)$ corresponding to C , under the map $H^0(Y, \mathcal{L}(p)) \rightarrow H^0(Y, \omega_Z(2+p-y))$ coming from diagram 4.

Remark 6.4. We may ask which locally free sheaves \mathcal{E} arise from this construction. Let us say that a locally free sheaf \mathcal{F} is normalized if it has a global section, while $\mathcal{F}(-1)$ has no section. Let \mathcal{F} be a locally free normalized \mathcal{O}_H -module of rank two. Then one may check that there exists a curve $C \subset 2H$ such that $\pi_*(\mathcal{I}_{C,2H})$ is isomorphic to $\mathcal{F}(n)$ for some integer n , if and only if $c_1\mathcal{F} \leq 1$. In particular, all unstable locally free sheaves of rank two arise from this construction.

Corollary 6.5. *Let C be a curve in $2H$ with $T(C) = \{Z, Y, P\}$. Then*

$$\begin{aligned} h^0(\mathcal{I}_C(n)) &= h^0(\mathcal{O}_{\mathbb{P}^3}(n-2)) + h^0(\mathcal{O}_H(n-y-1)) + \\ &\quad + h^0(\mathcal{I}_{Z,H}(n-p)) \end{aligned}$$

and

$$\begin{aligned} h^2(\mathcal{I}_C(n)) &= h^0(\mathcal{O}_{\mathbb{P}^3}(-n-4)) - h^0(\mathcal{O}_{\mathbb{P}^3}(-n-2)) + \\ &\quad + h^0(\mathcal{O}_H(p-3-n)) + h^0(\mathcal{I}_{Z,H}(y-2-n)). \end{aligned}$$

Proof. The formula for the postulation follows immediately from the exact sequence (2). To obtain the formula for h^2 we compute $h^2\mathcal{E}(n)$ using Serre duality:

$$h^2\mathcal{E}(n) = h^0\mathcal{E}(p+y-2-n) = h^0\mathcal{O}_H(p-3-n) + h^0\mathcal{I}_{Z,H}(y-2-n).$$

\square

We can also determine the postulation character of a curve C in $2H$. Recall [12] for any numerical function $f(n)$ one defines the difference function $\partial f(n) = f(n) - f(n-1)$. If C is a curve in \mathbb{P}^3 , one defines its *postulation character* γ_C by

$$\gamma_C(n) = \partial^3(h^0(\mathcal{I}_C(n)) - h^0(\mathcal{O}_{\mathbb{P}^3}(n))).$$

Similarly, for a zero-dimensional closed subscheme $Z \subset \mathbb{P}^2$ we define its postulation character

$$\gamma_Z(n) = \partial^2(h^0(\mathcal{I}_Z(n)) - h^0(\mathcal{O}_{\mathbb{P}^2}(n))).$$

By analogy with the case of ACM curves [12], I.2 and V.1.3, one shows easily the following:

Proposition 6.6. *Let Z be a zero dimensional closed subscheme of \mathbb{P}^2 of degree z , and let s be the least degree of a curve containing Z . Then*

$$\gamma_Z(n) = \begin{cases} 0 & \text{if } n < 0, \\ -1 & \text{if } 0 \leq n < s, \\ a_n \geq 0 & \text{if } n \geq s. \end{cases}$$

Furthermore, γ_Z is a character, so $\sum_{n \geq s} a_n = s$, and we can determine the degree z by

$$z = \sum_{n \geq s} n a_n - \frac{1}{2} s(s-1).$$

Conversely, given integers $s \geq 1$ and $a_n \geq 0$ for $n \geq s$ there exists a reduced zero-dimensional closed subscheme $Z \subset \mathbb{P}^2$ with postulation character as above.

Now using (6.5) we can compute the postulation character of C . We find

Theorem 6.7. *Let C be a curve in $2H$ with associated triple $\{Z, Y, P\}$. Then there is an integer s with $0 \leq s \leq y$ and there are integers b_n for $n \geq p + s$ such that $\sum b_n = s$, and the postulation character γ_C is the sum of the following functions: -1 in degree 0, -1 in degree 1, 1 in degree $y + 1$, 1 in degree $p + s$, and $\partial\beta$, where*

$$\beta(n) = \begin{cases} 0 & \text{if } n < p + s \\ b_n & \text{if } n \geq p + s. \end{cases}$$

Conversely, given integers $0 \leq s \leq y \leq p$ and $b_n \geq 0$ for $n \geq p + s$ with $\sum b_n = s$, there exists a curve C in $2H$ with postulation character as described.

Remark 6.8. In the notation of (6.7), the degree z of Z is determined by

$$z = \sum_{n \geq p+s} (n-p)b_n - \frac{1}{2} s(s-1).$$

Remark 6.9. One can see easily that the possible postulation characters of curves on surfaces of degree 2 other than $2H$ form a subset of the postulation characters described in (6.7). Thus we have found all postulation characters of curves in surfaces of degree 2. This gives some hope that one day it will be possible to determine all possible postulation characters of curves in \mathbb{P}^3 .

7. LIAISON

In this section we use notation and terminology of [7], section 4. The following proposition describes the behaviour of the triple $T(C) = \{Z, Y, P\}$ under liaison.

Proposition 7.1. *Let C be a curve in $2H$ with $T(C) = \{Z, Y, P\}$. Suppose S is a surface containing C and meeting H properly. Let $Q = H \cap S$. Then the curve D linked to C by the complete intersection $2H \cap S$ has triple $T(D) = \{Z, Q - P, Q - Y\}$; in particular, $Z \subset Q - P$.*

Conversely, if $Q \subset H$ is a curve containing P such that $Z \subset Q - P$, then C is linked by the complete intersection of $2H$ with some surface S to a curve D with $T(D) = \{Z, Q - P, Q - Y\}$.

In particular, Z and $P - Y$ are invariant under liaison on $2H$.

Proof. Suppose first C is linked to D by the complete intersection $E = 2H \cap S$, and let $T(D) = \{Z', Y', P'\}$. Let Q be the intersection of S with the reduced plane H . We claim that $Z' = Z$, $Y' = Q - P$ and $P' = Q - Y$.

By [7] page 317 the ideal sheaf of D in $2H$ is $\mathcal{H}om_{\mathcal{O}_{2H}}(\mathcal{I}_{C,2H}, \mathcal{I}_{E,2H})$ with its natural embedding in $\mathcal{H}om_{\mathcal{O}_{2H}}(\mathcal{I}_{E,2H}, \mathcal{I}_{E,2H}) \cong \mathcal{O}_{2H}$. By [6] exercises III 6.10 and 7.2, applying the functor $\mathcal{H}om_{\mathcal{O}_{2H}}(-, \mathcal{I}_{E,2H})$ to the exact sequence

$$0 \rightarrow \mathcal{I}_{Y,H}(-1) \xrightarrow{h} \mathcal{I}_{C,2H} \rightarrow \mathcal{I}_{Z,H}(-P) \rightarrow 0$$

we obtain a long exact sequence

$$0 \rightarrow \mathcal{I}_{Q-P,H}(-1) \xrightarrow{h} \mathcal{I}_{D,2H} \rightarrow \mathcal{I}_{Q-Y,H} \rightarrow \omega_Z(p - q + 2) \rightarrow 0.$$

As $\omega_Z(p - q + 2) \cong \mathcal{O}_Z(y - q)$, the kernel of the last map is $\mathcal{I}_{Z,H}(Y - Q)$, and our claim follows.

Conversely, suppose that Q is a curve in H containing P , and such that $Z \subset Q - P$. The exact sequence (2) tells us that there exists a surface S containing C but not H , whose intersection with H is Q . By what we have just shown, $T(D) = \{Z, Q - P, Q - Y\}$. \square

Corollary 7.2. *Let C be a curve in $2H$ with $T(C) = \{Z, Y, P\}$, and let Y' be a curve in H containing Z . Let y and y' be the degrees of Y and Y' respectively. There is curve D obtained from C by an elementary biliaison of height $y' - y$ on $2H$ with $T(D) = \{Z, Y', Y' + P - Y\}$.*

Proof. Use the above proposition with $Q = P + Y' = (Y' + P - Y) + Y$. \square

Corollary 7.3. *Let C be a curve in $2H$ with $T(C) = \{Z, Y, P\}$. Suppose that C is not arithmetically Cohen-Macaulay, that is, Z is not empty. Then C is minimal in its biliaison class if and only if Y has minimal degree among curves in H containing Z .*

Proof. If $h^0(H, \mathcal{I}_{Z,H}(\deg Y - 1)) > 0$, by the previous corollary there is a curve obtained from C by an elementary biliaison of negative height, hence C is not minimal.

If $h^0(H, \mathcal{I}_{Z,H}(\deg Y - 1)) = 0$, then by corollary 6.5 we have

$$h^2(\mathbb{P}^3, \mathcal{I}_C(1)) - 2h^2(\mathbb{P}^3, \mathcal{I}_C) + h^2(\mathbb{P}^3, \mathcal{I}_C(-1)) \leq 1.$$

Now it follows from [12] Proposition III.3.5 that C is minimal: see [19] Corollary 4.4. \square

8. CONNECTEDNESS OF THE HILBERT SCHEME

Theorem 8.1. *The Hilbert scheme $H_{d,g}(2H)$ is connected.*

Proof. If $g = \frac{1}{2}(d-1)(d-2)$ and $d \neq 2$, $H_{d,g}$ is irreducible by theorem 5.1.

To handle the case $d = 2$ and $g = 0$, we fix homogeneous coordinates $[x : y : z : w]$ on \mathbb{P}^3 , so that $x = 0$ is an equation for H , and we look at the family of curves C_t in $\mathbb{P}^3 \times \text{Spec } k[t]$ defined by the global ideal

$$I = \langle x^2, xy, y^2, x + ty \rangle.$$

For $t \neq 0$, C_t is in $H_{0,1,1}$, while C_0 belongs to $H_{0,0,2}$. It now follows from remark 5.2 that $H_{2,0}$ is connected.

If $g < \frac{1}{2}(d-1)(d-2)$, by remark 5.2 we have set theoretically

$$H_{d,g} = \bigcup_{1 \leq y \leq y_M} H_{z_y, y, d-y}$$

where

$$z_y = \frac{1}{2}(d-2)(d-3) - g - (y-1)(d-y-2).$$

In particular, $H_{d,g}$ is irreducible for $d \leq 3$, and for $d \geq 4$ the theorem is a consequence of the following proposition. \square

Proposition 8.2. *If $y \geq 2$, $p \geq y$ and $r \geq 0$, there is a curve in $H_{r,y,p}$ specializing to one in $H_{r+p-y,y-1,p+1}$.*

Proof. Suppose the claim is true when $y = 2$. Then by adding $y - 2$ times a plane section (i.e. by performing an elementary biliaison of height $y - 2$ on $2H$, see [10] proposition 1.6) we find that the claim is true for all $y \geq 2$, and we are done.

We now construct a family of curves in $H_{r,2,p}$ specializing to one in $H_{r+p-2,1,p+1}$. This has already been done by Nollet in [16], example 3.10, in the case $p = 2$ and $r = 1$, and his construction, generalizes without any major modification (Nollet noticed this independently [15]). Here are the details.

As above we fix homogeneous coordinates $[x : y : z : w]$ on \mathbb{P}^3 , so that $x = 0$ is an equation for H . We let Y and P denote respectively the double line $x = y^2 = 0$ and the multiple line $x = y^p = 0$. Let Z have equations $x = y = f = 0$ where f is a form of degree $r = \deg Z$ in $k[z, w]$.

To give a curve C in $2H$ with $T(C) = \{Z, Y, P\}$ is by proposition 3.1 the same as giving a morphism $v : \mathcal{I}_{Z,H}(-p) \rightarrow \mathcal{O}_Y(-1)$ whose image in $H^0(\omega_Z(p))$ generates $\omega_Z(p)$ at every point. This amounts to choosing forms s and g in $k[z, w]$, of degrees $p - 1$ and $r + p - 2$ respectively, and such that g and f have no common zeros on the line $x = y = 0$. The corresponding morphism $\mathcal{I}_{Z,H}(-p) \rightarrow \mathcal{O}_Y(-1)$ sends y to ys and f to $fs + yg$. For the corresponding curve C we have:

$$I_C = \langle x^2, xy^2, y^{p+1} + xys, xfs + xyg + y^p f \rangle.$$

Now we look at the family of curves in $\mathbb{P}^3 \times \text{Spec } k[t]$ obtained by flattening the ideal generated by x^2, A, B, C where

$$\begin{aligned} A &= xy^2 \\ B &= ty^{p+1} - xyz^{p-1} \\ C &= xyw^{r+p-2} - tz^r(ty^p - xz^{p-1}). \end{aligned}$$

For $t \neq 0$ we obtain a curve C as above with $s = -\frac{1}{t}z^{p-1}$, $g = w^{r+p-2}$ and $f = -t^2z^r$. To see what happens at $t = 0$, we set

$$\begin{aligned} D &= \frac{1}{t}(w^{r+p-2}B + z^{p-1}C) = y^{p+1}w^{r+p-2} + xz^{r+2p-2} + tF \\ E &= \frac{1}{t}(z^{p-1}A + yB) = y^{p+2}. \end{aligned}$$

It follows that the ideal of the limit scheme C_0 contains the ideal

$$J = \langle x^2, xy^2, xyz^{p-1}, xyw^{r+p-2}, y^{p+2}, y^{p+1}w^{r+p-2} + xz^{r+2p-2} \rangle.$$

The saturation of J is the ideal

$$I = \langle x^2, xy, y^{p+2}, y^{p+1}w^{r+p-2} + xz^{r+2p-2} \rangle.$$

But this is the homogeneous ideal of a curve D in the double plane: $Y(D)$ is the line $x = y = 0$, $P(D)$ has equations $x = y^{p+1} = 0$, and $Z(D)$ is defined on $Y(D)$ by the equation $w^{r+p-2} = 0$. Hence D belongs to $H_{r+p-2,1,p+1}$. In particular, D has the same degree and genus as C_0 . So we must have $D = C_0$, and this finishes the proof.

We remark that in this family the zero dimensional scheme Z associated to C_t is supported at the point $[0 : 0 : 1 : 0]$ for $t \neq 0$, and at the point $[0 : 0 : 0 : 1]$ for $t = 0$! \square

9. EXTREMAL AND SUBEXTREMAL CURVES

Given a curve C , the function $\rho_C : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $\rho_C(n) = h^1 \mathcal{I}_C(n)$ is called the Rao function of C . It is the Hilbert function of the Rao module $M_C = H_*^1(\mathbb{P}^3, \mathcal{I}_C)$ of C .

Theorem 9.1 ([13],[17]). *Let $C \subset \mathbb{P}^3$ be a curve of degree d and arithmetic genus g . Then*

1. *C is planar if and only if $g = \frac{1}{2}(d-1)(d-2)$.*
2. *If C is not planar, then $d \geq 2$, $g \leq \frac{1}{2}(d-2)(d-3)$ and*

$$\rho_C(n) \leq \rho_{d,g}^E(n) \quad \text{for all } n \in \mathbb{Z}$$

where

$$\rho_{d,g}^E(n) = \begin{cases} \max(0, \rho_{d,g}^E(n+1) - 1) & \text{if } n \leq -1, \\ \frac{1}{2}(d-2)(d-3) - g & \text{if } 0 \leq n \leq d-2, \\ \max(0, \rho_{d,g}^E(n-1) - 1) & \text{if } n \geq d-1. \end{cases}$$

Definition 9.2. A curve $C \subset \mathbb{P}^3$ of degree d and genus g is called *extremal* if it is not planar and $\rho_C = \rho_{d,g}^E$.

Remark 9.3. The definition of an extremal curve in [14], [17] is different from ours, since they require the curve not to be ACM, but we allow the ACM case if $\rho_{d,g}^E = 0$.

Remark 9.4. There exist extremal curves (for example in $2H$, see below) if $d = 2$ and $g \leq -1$ and if $d \geq 3$ and $g \leq \frac{1}{2}(d-2)(d-3)$.

The following characterization of extremal curves is essentially contained in [14] and [17].

Proposition 9.5. *Let C be a curve of degree $d \geq 2$ and genus g . The following are equivalent:*

1. C is extremal;
2. either C is a minimal curve contained in the union of two distinct planes, or C is obtained from a plane curve by an elementary biliaison of height one on a quadric surface;
3. either C is ACM with $(d, g) \in \{(3, 0), (4, 1)\}$ and C is contained in an integral quadric surface, or C is a non-planar curve which contains a plane curve of degree $d - 1$.

Proof. Looking at [13], [17] one sees that C is extremal if and only if it is not planar and $h^0\mathcal{I}_C(2) \geq 2$. From this we deduce that 1 implies 3. Curves C in the union of two distinct planes $H_1 \cup H_2$ are studied in [7], section 5: if $C \cap H_1$ has degree $d - 1$, then C is either minimal or ACM. From this we see that 3 implies 2.

Finally, we prove 2 implies 1. If C is obtained from a plane curve by an elementary biliaison of height one on a quadric surface, then one computes $g = \frac{1}{2}(d - 2)(d - 3)$ and so C is extremal. If C is a minimal curve in $H_1 \cup H_2$, then by [7], section 5, adding a suitable number (namely, $\frac{1}{2}(d - 2)(d - 3) - g - 1$) of plane sections to C we obtain a curve linearly equivalent to the disjoint union of two plane curves. Thus the Rao module of C is isomorphic to $R/(h, k, f, g)(a - 1)$ where $a = \frac{1}{2}(d - 2)(d - 3) - g$, R is the homogeneous coordinate ring of \mathbb{P}^3 , h, k are the equations of H_1 and H_2 , and f, g are forms of degrees $a, d + a - 2$ respectively, having no common zeros along the line $h = k = 0$. Hence $\rho_C = \rho_{d,g}^E$. \square

Theorem 9.6 ([17]). *Let $C \subset \mathbb{P}^3$ be a curve of degree d and genus g , which is neither planar nor extremal. Then $d \geq 3$ and*

1. $g \leq \frac{1}{2}(d - 3)(d - 4) + 1$; if equality holds, then $d \geq 5$ and C is ACM;
2. if $d \geq 4$, then

$$\rho_C(n) \leq \rho_{d,g}^S(n) \quad \text{for all } n \in \mathbb{Z}$$

where

$$\rho_{d,g}^S(n) = \begin{cases} \max(0, \rho_{d,g}^S(n + 1) - 1) & \text{if } n \leq 0, \\ \frac{1}{2}(d - 3)(d - 4) + 1 - g & \text{if } 1 \leq n \leq d - 3, \\ \max(0, \rho_{d,g}^S(n - 1) - 1) & \text{if } n \geq d - 2. \end{cases}$$

Definition 9.7. A curve $C \subset \mathbb{P}^3$ of degree $d \geq 4$ and genus g is called *subextremal* if it is neither planar nor extremal and $\rho_C = \rho_{d,g}^S$. Again, this differs from the terminology of [17] in that we include among

subextremal curves those ACM curves of degree $d \geq 5$ which have genus $1/2(d-3)(d-4) + 1$.

Remark 9.8. There exist subextremal curves (for example in $2H$, see below) if $d = 4$ and $g \leq 0$ and if $d \geq 5$ and $g \leq \frac{1}{2}(d-3)(d-4) + 1$.

Proposition 9.9. *Let $C \subset \mathbb{P}^3$ be a curve of degree $d \geq 4$ and genus g . The following are equivalent:*

1. C is subextremal;
2. C is obtained from an extremal curve by an elementary biliaison of height one on a quadric surface;
3. either C is ACM and $(d, g) \in \{(5, 2), (6, 4)\}$, or C is a divisor of type $(1, 3)$ on a smooth quadric surface, or there is a plane H such that $\mathcal{I}_{C \cap H, H} = \mathcal{I}_{Z, H}(2 - d)$ with Z zero-dimensional contained in a line, and the residual scheme $\text{Res}_H(C)$ to the intersection of C with H is a plane curve of degree two.

Proof. If C is not ACM, the equivalence of 1 and 2 is the content of Theorem 2.14 in [17]. The same proof works for the ACM case as well (cf. [17], Lemma 2.5).

We claim that 2 implies 3: suppose C is obtained from an extremal curve B by an elementary biliaison of height one on a quadric surface F . If there is a plane H whose intersection with B has degree $d - 3 = \deg B - 1$ and if F contains H as a component, then $\mathcal{I}_{C \cap H, H} = \mathcal{I}_{Z, H}(2 - d)$ with Z zero-dimensional contained in a line, and $\text{Res}_H(C)$ is a plane curve. Otherwise, by proposition 9.5 either B is an ACM curve with $(d, g) \in \{(3, 0), (4, 1)\}$, or B is a divisor of type $(0, 2)$ on the smooth quadric surface F . So C is either ACM with $(d, g) \in \{(5, 2), (6, 4)\}$, or C is a divisor of type $(1, 3)$ on F .

Finally, we show that 3 implies 1. The special cases are clear, so we may assume that there is a plane H such that $\mathcal{I}_{C \cap H, H} = \mathcal{I}_{Z, H}(2 - d)$ with Z zero-dimensional contained in a line, and $Y = \text{Res}_H(C)$ is a plane curve of degree two. Then there is an exact sequence:

$$(6) \quad 0 \rightarrow \mathcal{I}_{Y, \mathbb{P}^3}(-1) \xrightarrow{h} \mathcal{I}_{C, \mathbb{P}^3} \rightarrow \mathcal{I}_{Z, H}(2 - d) \rightarrow 0$$

from which we deduce that $\text{length } Z = \frac{1}{2}(d-3)(d-4) + 1 - g$, that C is contained in a unique quadric surface and that $\rho_C(n) = \rho_{d, g}^S(n)$ for $n \geq 1$. Since the Rao function of a curve of degree d contained in a quadric surface is symmetric around $\frac{d-2}{2}$, C is subextremal. \square

As a corollary, we now identify extremal and subextremal curves in a smooth quadric surface and in a double plane, leaving the case of the quadric cone and of the union of two distinct planes to the reader.

Corollary 9.10. *Let C be an effective divisor of type (a, b) on the smooth quadric surface Q , with $a \leq b$. Then*

1. C is planar if and only if $(a, b) \in \{(0, 1), (1, 1)\}$;
2. C is extremal if and only if $(a, b) \in \{(0, 2), (1, 2), (2, 2)\}$;
3. C is subextremal if and only if $(a, b) \in \{(1, 3), (2, 3), (3, 3)\}$.

Corollary 9.11. *Let C be a curve in the double plane $2H$ with associated triple $\{Z, Y, P\}$. Then*

1. C is planar if and only if either $y = 0$, or $p = y = 1$ and $z = 0$;
2. C is extremal if and only if either $y = 1$ and $p \geq 2$, or $y = p = 1$ and $z \geq 1$, or $y = p = 2$ and $z = 0$;
3. C is subextremal if and only if Z is contained in a line and either $y = 2$, $p \geq 3$, or $y = p = 2$ and $z \geq 1$, or $y = p = 3$ and $z = 0$.

The following proposition has been proven independently by Nollet [15]; see also [1], especially Proposition 4.15.

Proposition 9.12. *Let H be a plane in \mathbb{P}^3 . For every $d \geq 5$ and $g \leq \frac{1}{2}(d-3)(d-4)+1$ (resp. $d = 4$ and $g \leq 0$) the closure of the family of subextremal curves in $H_{d,g}(2H)$ contains an extremal curve. In particular, for $d \geq 4$ and $g \geq \frac{1}{2}(d-3)(d-4)+1$ the Hilbert scheme $H_{d,g}(\mathbb{P}^3)$ is connected.*

Proof. The first statement follows from proposition 8.2. For $d \geq 4$, $g > \frac{1}{2}(d-3)(d-4)+1$ and for $(d, g) = (4, 1)$ the Hilbert scheme $H_{d,g}(\mathbb{P}^3)$ is irreducible [14], section 5. For $d \geq 5$ and $g = \frac{1}{2}(d-3)(d-4)+1$, $H_{d,g}(\mathbb{P}^3)$ has two irreducible components, namely the closure of the family of subextremal curves and the family of extremal curves [1]. Hence it is connected by the first statement. \square

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